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ON FINITE PURE BENDING OF CURVED TUBES†

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Abstract—The present paper, in essence, owes its origin to a statement in the original manuscript of a paper by Boyle‡ in which doubts were expressed concerning the recovery possibility of the results of finite-bending theory for straight tubes as a limiting case of the corresponding results for curved tubes. Although this statement does not occur in the final version of the manuscript, Boyle's using a state of rotational displacement with two scalar components as against the evident possibility of needing just one such component,, and his treating the problem as a sixth order problem without regard to its possible reduction to one fourth order via use of two first integrals, suggested to the author that the following discussion (with the additional benefit of an opportunity of correcting an earlier oversight in Refs.[11, 12] would serve a useful purpose.

INTRODUCTION

In what follows we complement an earlier approximate analysis of this problem[1], as well as an earlier exact analysis of Brazier's problem of bending instability of straight tubes[2, 3], by the formulation of a system of exact equations for the pure-bending problem of curved tubes, in such a way that the problem of the originally straight tube re-appears as a special case. Our interest in this question was re-awakened by doubts which have been expressed by Boyle on whether the finite-bending problem of the straight tube could in fact be recovered as a limiting case of the problem of the curved tube. An additional element of interest is thought to be that the derivation given here does at the same time serve as an example of application of a general non-linear shell theory which had earlier been formulated by Simmonds and Danielson[4, 5] and by the present writer[6], in supplementation of an earlier direct formulation by Axelrad[7] which extended the author's results on finite axi-symmetric deformations without circumferential bending[8].

GEOMETRY OF UNDEFORMED AND DEFORMED CURVED TUBE

We write as vector equation for the middle surface of the *undeformed* tube

$$\mathbf{r} = r(\xi)\mathbf{i}_r + z(\xi)\mathbf{i}_3, \tag{1}$$

where $\mathbf{i}_r = \mathbf{i}_1 \cos\theta + \mathbf{i}_2 \sin\theta$.

From eqn (1) we obtain as expressions for tangent vectors,

$$\mathbf{r}_{,\xi} = r'\mathbf{i}_r + z'\mathbf{i}_3, \quad \mathbf{r}_{,\theta} = r\mathbf{i}_\theta, \tag{2}$$

where $\mathbf{i}_\theta = -\mathbf{i}_1 \sin\theta + \mathbf{i}_2 \cos$, and then as expressions for tangent *unit* vectors

$$\mathbf{t}_\xi = \mathbf{i}_r \cos\phi + \mathbf{i}_3 \sin\phi, \quad \mathbf{t}_\theta = \mathbf{i}_\theta, \tag{3}$$

with $r' = \alpha\cos\phi$, $z' = \alpha\sin\phi$ and $\alpha \equiv \alpha_\xi = [(r')^2 + (z')^2]^{1/2}$ and $\alpha_\theta = r$.

The two tangent unit vectors in (3) are complemented by a normal unit vector

$$\mathbf{n} = \mathbf{t}_\theta \times \mathbf{t}_\xi = \mathbf{i}_r \sin\phi - \mathbf{i}_3 \cos\phi, \tag{4}$$

with differentiation formulas

$$\begin{aligned} &\mathbf{t}_{\xi,\xi} = -\phi'\mathbf{n}, \quad \mathbf{n}_{,\xi} = \phi'\mathbf{t}_\xi, \quad \mathbf{t}_{\theta,\xi} = 0, \\ &\mathbf{t}_{\xi,\theta} = \cos\phi\,\mathbf{t}_\theta, \quad \mathbf{n}_{,\theta} = \sin\phi\,\mathbf{t}_\theta, \quad \mathbf{t}_{\theta,\theta} = -\cos\phi\,\mathbf{t}_\xi - \sin\phi\,\mathbf{n}. \end{aligned} \tag{5}$$

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‡J. T. Boyle, The finite-bending of curved tubes. *Int. J. Solids Structures* 17, 515–529 (1981).

Equations (5), in conjunction with the corresponding general formulas in [9], imply as expressions for curvature radii

$$\frac{1}{R_\xi} = \frac{\phi'}{\alpha}, \quad \frac{1}{R_\theta} = \frac{\sin\phi}{r}, \quad \frac{1}{S} = -\frac{\cos\phi}{r}. \tag{6}$$

In order to describe a state of pure bending we write as vector equation for the middle surface of the *deformed* tube, as previously proposed in [9]

$$\boldsymbol{\rho} = (r+u)\mathbf{i}_\rho + (z+w)\mathbf{i}_3, \tag{7}$$

where $\mathbf{i}_\rho = \mathbf{i}_1 \cos(1+k)\theta + \mathbf{i}_2 \sin(1+k)\theta$ and $u = u(\xi)$, $w = w(\xi)$.

Expressions for tangent and normal unit vectors associated with the surface eqn (7) are written, in accordance with the notation used in [6], in the form

$$\mathbf{T}_\xi = \mathbf{i}_\rho \cos\Phi + \mathbf{i}_3 \sin\Phi, \quad \mathbf{T}_\theta = i_\Theta, \tag{8}$$

$$\mathbf{N} = \mathbf{i}_\rho \sin\Phi - \mathbf{i}_3 \cos\Phi, \tag{9}$$

with $\mathbf{i}_\Theta = -\mathbf{i}_1 \sin(1+k)\theta + \mathbf{i}_2 \cos(1+k)\theta$. Equations (8) and (9) imply the differentiation formulas

$$\mathbf{T}_{\xi,\xi} = -\Phi'\mathbf{N}, \quad \mathbf{N}_{,\xi} = \Phi'\mathbf{T}_\xi, \ \mathbf{T}_{\theta,\xi} = 0, \quad \mathbf{T}_{\xi,\theta} = (1+k)\cos\Phi\mathbf{T}_\theta,$$
$$\mathbf{N}_{,\theta} = (1+k)\sin\Phi\mathbf{T}_\theta, \quad \mathbf{T}_{\theta,\theta} = -(1+k)(\cos\Phi\mathbf{T}_\xi + \sin\Phi\,\mathbf{N}), \tag{10}$$

and these in turn imply as expressions for curvature radii

$$\frac{1}{r_\xi} = \frac{\Phi'}{\alpha}, \quad \frac{1}{r_\theta} = (1+k)\frac{\sin\Phi}{r}, \quad \frac{1}{s} = -(1+k)\frac{\cos\Phi}{r}. \tag{11}$$

STRAIN DISPLACEMENT RELATIONS

We define membrane and transverse shearing strain components ϵ_ξ, ϵ_θ and γ in accordance with the work in [5, 6] by means of the relations

$$\boldsymbol{\rho}_{,\xi} = \alpha[(1+\epsilon_\xi)\mathbf{T}_\xi + \gamma\mathbf{N}], \quad \boldsymbol{\rho}_{,\theta} = r(1+\epsilon_\theta)\mathbf{T}_\theta. \tag{12}$$

Equations (12), in conjunction with eqns (7)–(9), result in strain displacement relations of the form

$$\begin{aligned} \alpha\epsilon_\xi &= u'\cos\Phi + w'\sin\Phi + \alpha[\cos(\Phi-\phi) - 1], \\ \alpha\gamma &= i'\sin\Phi - w'\cos\Phi + \alpha\sin(\Phi-\phi), \\ r\epsilon_\theta &= (1+k)u - kr. \end{aligned} \tag{13}$$

We define bending strain couples κ_ξ and κ_θ on the basis of eqns (6) and (11), again in accordance with [5, 6], in such a way that

$$\alpha\kappa_\xi = \Phi' - \phi', \quad r\kappa_\theta = (1+k)\sin\Phi - \sin\phi. \tag{14a}$$

An additional component λ which, in general, is not of physical significance is given by

$$-r\lambda = (1+k)\cos\Phi - \cos\phi. \tag{14b}$$

For what follows it is of importance that the three components in (13) are such as to imply a compatibility equation of the form

$$(r\epsilon_\theta)' - (1+k)\alpha(\epsilon_\xi\cos\Phi + \gamma\sin\Phi) = \alpha[(1+k)\cos\Phi - \cos\phi], \tag{15}$$

with the corresponding relation in [6] written in terms of κ_θ and λ, rather than in terms of Φ.

Equations (13)–(15), for the special case $k = 0$, are equivalent to relations which have previously been obtained by direct considerations[10].

EQUILIBRIUM EQUATIONS

Consistent with the notation in [8] we have as expressions for stress resultant vectors

$$\mathbf{N}_\xi = N_\xi \mathbf{T}_\xi + Q\mathbf{N}, \quad \mathbf{N}_\theta = N_\theta \mathbf{T}_\theta, \tag{16}$$

and, with $\mathbf{N} \times \mathbf{T}_\xi = -\mathbf{T}_\theta$ and $\mathbf{N} \times \mathbf{T}_\theta = \mathbf{T}_\xi$, as expressions for stress couple vectors

$$\mathbf{M}_\xi = -M_\xi \mathbf{T}_\theta, \quad \mathbf{M}_\theta = M_\theta \mathbf{T}_\xi. \tag{17}$$

Equations of equilibrium are, in terms of these

$$(r\mathbf{N}_\xi)_{,\xi} + (\alpha \mathbf{N}_\theta)_{,\theta} = 0, \tag{18}$$

$$(r\mathbf{M}_\xi)_{,\xi} + (\alpha \mathbf{M}_\theta)_{,\theta} + \boldsymbol{\rho}_{,\xi} \times r\mathbf{N}_\xi + \boldsymbol{\rho}_{,\theta} \times \alpha \mathbf{N}_\theta = 0. \tag{19}$$

Equation (18) may be shown, with the help of (8), (9) and (12), to be equivalent to two scalar equations of force equilibrium in *axial* and *radial* direction,

$$[r(N_\xi \sin \Phi - Q \cos \Phi]' = 0, \tag{20a}$$

$$[r(N_\xi \cos \Phi + Q \sin \Phi]' - (1 + k)\alpha N_\theta = 0, \tag{20b}$$

and eqn (19) reduces to one scalar equation of moment equilibrium

$$(rM_\xi)' - (1 + k)\alpha \cos \Phi M_\theta - \alpha r[(1 + \epsilon_\xi)Q - \gamma N_\xi] = 0. \tag{21}$$

Having (20b) we see that in fact $\oint N_\theta \alpha \, d\xi = 0$, as it should be.† The corresponding equation of overall moment equilibrium can then be written in the form

$$M\mathbf{i}_3 = \oint (\mathbf{M}_\theta + \boldsymbol{\rho} \times \mathbf{N}_\theta)\alpha \, d\xi. \tag{22}$$

With $\mathbf{M}_\theta$ as in (17) and with $\mathbf{N}_\theta$ as in (16) and (20b), and upon integration by parts after introducing N_θ from (20b), it is found that (22) is equivalent to the relation

$$\begin{aligned} M\mathbf{i}_3 = &\oint M_\theta(\mathbf{i}_3 \sin \Phi + \mathbf{i}_\rho \cos \Phi)\alpha \, d\xi \\ &+ \oint r \frac{N_\xi \cos \Phi + Q \sin \Phi}{1 + k} \{[(1 + \epsilon_\xi) \sin \Phi - \gamma \cos \Phi]\mathbf{i}_\rho \\ &- [(1 + \epsilon_\xi) \cos \Phi + \gamma \sin \Phi]\mathbf{i}_3\}\alpha \, d\xi. \end{aligned} \tag{23}$$

Equation (23) is evidently equivalent to two scalar equations. One of these will serve to determine one of the constants of integration in the solution of the differential equations of the problem. Evaluation of the other scalar equation, involving the cross sectional moment M, will lead to the desired moment-curvature change relation for the tube.

CONSTITUTIVE EQUATIONS

A representative system of such relations is

$$M_\xi = D_\xi \kappa_\xi + D_\nu \kappa_\theta, \quad M_\theta = D_\theta \kappa_\theta + D_\nu \kappa_\xi, \tag{24}$$

$$\epsilon_\xi = B_\xi N_\xi - B_\nu N_\theta, \quad \epsilon_\theta = B_\theta N_\theta - B_\nu N_\xi \tag{25}$$

$$\gamma = B_t Q, \tag{26}$$

†This statement is readily modified so as to apply directly to open cross section tubes as well.

where we may note the possibility of more general relations, so as to describe the behavior of laminated polar-orthotropic tubes as well.

REDUCTION TO TWO SIMULTANEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

Considering the known possibility of the reduction of the linear problem of the curved tube[11, 12] as well as of the non-linear problem of the straight tube[2] to a system of two second order differential equations, it suggests itself that the relation between the non-linear problem of the straight tube and the non-linear problem of the curved tube may be most readily seen by reducing the latter problem to such a system.

Analogous to what happens for the known special case, the two simultaneous differential equations are obtained upon appropriate substitutions in the compatibility equation (15) and the moment equilibrium eqn (21), with the basic dependent variables being rotational displacement function Φ and a stress function Ψ, defined by

$$(1+k)\Psi = rN_\xi \cos\Phi + rQ\sin\Phi. \tag{27}$$

With (27) and (20a, b) we may write as expressions for stress resultants

$$\alpha N_\theta = \Psi' \tag{28a}$$

$$rN_\xi = (1+k)\Psi\cos\Phi + C\sin\Phi, \tag{28b}$$

$$rQ = (1+k)\Psi\sin\Phi - C\cos\Phi, \tag{28c}$$

and the consequences of the overall vector equilibrium eqn (23) become

$$M = \oint\{M_\theta\sin\Phi - [(1+\epsilon_\xi)\cos\Phi + \gamma\sin\Phi]\Psi\}\alpha\,d\xi, \tag{29}$$

$$0 = \oint\{M_\theta\cos\Phi + [(1+\epsilon_\xi)\sin\Phi - \gamma\cos\Phi]\Psi\}\alpha\,d\xi. \tag{30}$$

In order to simplify the analysis from here on, without loosing sight of primary effects, we will assume now a limiting form of the constitutive eqns (24)–(26) setting

$$\epsilon_\xi = \gamma = 0,\ \ M_\theta = 0, \tag{31}$$

and

$$M_\xi = D\kappa_\xi,\ \ \epsilon_\theta = BN_\theta. \tag{32}$$

The first of the two simultaneous differential equations follows now from (15) in the form

$$\frac{1}{\alpha}\left(rB\frac{\Psi'}{\alpha}\right)' = (1+k)\cos\Phi - \cos\phi. \tag{33}$$

The second differential equation follows from (21) in the form

$$\frac{1}{\alpha}\left(rD\frac{\Phi'-\phi'}{\alpha}\right)' = (1+k)\Psi\sin\Phi - C\cos\Phi. \tag{34}$$

Conditions for the determination of the four constants of integration for the solution of (33) and (34), and for the fifth constant C, are eqn (30), in conjunction with conditions of single valuedness for Φ, Φ', Ψ and Ψ'.†

†We use this opportunity to note that the assumption $C = 0$ in [11, 12], without explicit consideration of eqn (30), represents an approximation rather than an exact result, with this, however, being of no consequence for the quantitative data in [11].

EQUATIONS FOR UNIFORM TUBES OF CIRCULAR CROSS SECTION

The case of a tube with circular cross section is given upon setting

$$r = a + b\sin\xi,\ z = -b\cos\xi \tag{35}$$

and therewith

$$\alpha = b,\ \phi = \xi. \tag{36}$$

The significance of the deformational parameter k is established by noting that due to deformation the radius a of the tube will change to a radius ρ, in such a way that the length of the centerline of the tube remains unchanged. This means that we have

$$a\theta = \rho(1+k)\theta, \tag{37}$$

and therewith as expression for curvature change

$$K \equiv \frac{1}{\rho} - \frac{1}{a} = \frac{k}{a}. \tag{38}$$

In rewriting the differential equations (33) and (34) for the case that eqns (35) and (36) hold it is convenient to introduce the notation

$$\Phi - \phi = \beta. \tag{39}$$

Furthermore, it will be assumed for simplicity's sake that B and D are constants. Therewith, and with a useful rearrangement of the terms on the right of (33), eqns (33) and (34) take on the form

$$\frac{aB}{b^2}\left[\left(1+\frac{b}{a}\sin\xi\right)\Psi'\right]' = k\cos\xi + (1+k)[\cos(\beta+\xi) - \cos\xi], \tag{40}$$

$$\frac{aD}{b^2}\left[\left(1+\frac{b}{a}\sin\xi\right)\beta'\right]' = (1+k)\Psi\sin(\beta+\xi) - C\cos(\beta+\xi). \tag{41}$$

As regards the four boundary conditions of single-valuedness, these may be replaced, because of the symmetry of the cross section, by the four conditions

$$\xi = \pm\frac{1}{2}\pi;\ \beta = 0,\ \Psi = 0, \tag{42}$$

with β and Ψ to be determined in the interval $-\pi/2 \le \xi \le \pi/2$. At the same time, eqns (29) and (30), with (31), may be rewritten in the form

$$-2b\int_{-\pi/2}^{\pi/2} \Psi\cos(\beta+\xi)\,d\xi = M, \tag{43}$$

$$\int_{-\pi/2}^{\pi/2} \Psi\sin(\beta+\xi)\,d\xi = 0, \tag{44}$$

with eqn (44) serving to determine the value of C.

In order to see that the problem as stated by means of eqns (40)–(44) includes as limiting cases both the correct formulation of the linear problem of curved-tube bending[12] and the non-linear problem of Brazier-type instability of straight tubes[9] we introduce two new dependent variables f and g and a new constant c by means of the relations

$$\Psi = \Psi_0 f,\quad \beta = \beta_0 g,\quad C = \Psi_0 c, \tag{45}$$

where

$$\Psi_0 = \frac{Kb^2}{B}, \quad \beta_0 = \frac{Kb^2}{\sqrt{(BD)}}. \tag{46}$$

Therewith, and with the dimensionless parameter

$$\mu = \frac{b^2}{a\sqrt{(BD)}}, \tag{47}$$

eqns (40) and (41) may be written in the form

$$\left[\left(1+\frac{b}{a}\sin\xi\right)f'\right]' + (\mu+\beta_0)\left[\sin\xi\,\frac{\sin\beta_0 g}{\beta_0} + \cos\xi\,\frac{1-\cos\beta_0 g}{\beta_0}\right] = \cos\xi, \tag{48}$$

and

$$\left[\left(1+\frac{b}{a}\sin\xi\right)g'\right]' - (\mu+\beta_0)[\sin\xi\cos\beta_0 g + \cos\xi\sin\beta_0 g]f$$
$$= \mu c[\cos\xi\cos\beta_0 g - \sin\xi\sin\beta_0 g], \tag{49}$$

with boundary conditions $f(\pm\pi/2) = g(\pm\pi/2) = 0$, and with eqns (43) and (44) becoming

$$M = -2\sqrt{\left(\frac{B}{D}\right)}\,b\beta_0 \int_{-\pi/2}^{\pi/2} [\cos\xi\cos\beta_0 g - \sin\xi\sin\beta_0 g]f\,d\xi, \tag{50}$$

and

$$\int_{-\pi/2}^{\pi/2} [\sin\xi\cos\beta_0 g + \cos\xi\sin\beta_0 g]f\,d\xi = 0. \tag{51}$$

From this we recover the equations for Brazier type instability of straight tubes, in the form considered in [2, 3], by setting $\mu = 0$ and $b/a = 0$, and we recover the equations of linear curved-tube bending theory, as considered in [11, 12], by limiting ourselves to the parameter value range $|\beta_0| \ll 1$.

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